On the Asymptotic Number of Active Links in a Random Network

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A network of n transmitters and n receivers is considered. We assume that transmitter i aims to send data to its designated destination, receiver i. Communications occur in a single-hop fashion and destination nodes are simple linear receivers without multi-user detection. Therefore, in each time slot every source node can only talk to one other destination node. Thus, there is a total of n communication links. An important question now arises. How many links can be active in such a network so that each of them supports a minimum rate R_{min}? This dissertation is devoted to this problem and tries to solve it in two different settings, *dense* and *extended* networks. In both settings our approach is asymptotic, meaning, we only examine the behaviour of the network when the number of nodes tends to infinity. We are also interested in the events that occur asymptotically almost surely ($a.a.s.$), i.e., events that have probabilities approaching one as the size of the networks gets large. In the first part of the thesis, we consider a *dense network* where fading is the dominant factor affecting the quality of transmissions. Rayliegh channels are used to model the impact of fading. It is shown that $a.a.s.$ $\log(n)^2$ links can simultaneously maintain R_{min} and thus be active. In the second part, an *extended network* is considered where nodes are distant from each other and thus, a more complete model must take inter-node distances into account. We will show that in this case, almost all of the links can be active while maintaining the minimum rate.

Acknowledgements

First and most I am grateful to my supervisor Professor Ravi Mazumdar who was a full supporter during my Masters studies at Waterloo and from whom I learned a lot. His vast knowledge and insight into problems always encouraged me to try harder. I also benefited from excellent classes here at UofWaterloo among which stochastic calculus, offered by Prof. Heunis was an unforgettable exposure to rigorous probability theory. I must also thank all my friends here who made my stay at this small town an amazing memory. Abbas, Omid, Ehsan, Mohammad and Morteza, thank you all for your presence and for the beautiful moments you created for me. May we remain friends forever.

Dedication

To my parents, Amir and Nasrin who are God's greatest gift to me and to whom I always remain indebted.

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Chapter 1

Introduction and Preliminaries

During recent decades wireless communication has become one of the most important and, at the same time, challenging inventions of all the times. Around this technology, a huge amount of theoretical and practical research has been conducted to provide reliable and efficient data delivery. As other technologies progress, new demands emerge and more sophisticated problems must be solved. To address different needs, different types of wireless systems have been invented. In the following we will take a look at basic definition and different types of wireless networks.

1.1 Wireless Networks

A wireless network is a network of transmitters and receivers in which each transmitter (source node) aims to send a message (voice, data, multimedia) to a designated receiver (destination). According to the coverage area, underlying backbone and method of transmission, wireless networks are divided into different categories. Analysis and design of these structures, even in their simplest forms is among the hardest problems in communication theory. There are many problems yet to be solved, and one of the most important ones is the problem of capacity, i.e. the maximum data rate that can be delivered in the network. This is actually the main concern of this work. It is interesting that as the number of nodes in the network increases, it becomes more tractable and easier to analyse. Therefore, throughout this dissertation we will measure network's quantities asymptotically, meaning, when the size of the network tends to infinity. This enables us to exploit the powerful machinery of the large deviations theory. On the other hand, we will be dealing with events that occur asymptotically almost surely ($a.a.s.$) i.e., with a probability that tends to one as the number of nodes in the networks approaches infinity. This deterministic method has also been exploited in [3, 2].

The most obvious advantage of wireless technology is its freedom from messy bunches of wire that run here and there. This reduces costs associated with installation and maintenance. Technologies like bluetooth and WiFi can provide instantaneous communication without physical connection and set up. More importantly, wireless communication is making the idea of intelligent homes and devises a reality, a home where all of the appliances can communicate and share necessary information with each other and you. Another appealing factor of wireless systems is their flexibility, i.e. services can reach you wherever you are, all the time. This includes a very wide range from medical and health care to transportation and navigation services. Global coverage provided by wireless systems is the other advantage that makes their use appealing. With the aim of wireless technology, communication can reach where wiring is infeasible or costly, e.g. rural and sparse areas, battlefields, moving vehicles and outer space.

This long list of applications and benefits, explains the increasing dependence on wireless technology for business and personal use. However, like any other technological product, wireless communication brings with itself a range of problems and challenges that must be resolved. Among these are the following. The need for efficient hardware design: batteries need to live longer and, less power-hungry and faster hardware should be designed every day. Today, more and more communication must be accommodated in a limited radio spectrum. Therefore, efficient use of finite radio spectrum is an important issue. Cellular frequency reuse and multiple- input-multiple-output (MIMO) systems must be deployed in telecommunication systems to overcome this problem. Integrated services that combine voice, data and multimedia are getting more and more popular. A huge amount of data must be carried over a single medium with limited frequency band. This had been the motivation of many multiplexing techniques. In many applications, only certain amounts of delay, packet loss, Bit Error Rate (BER) are tolerable. maintaining quality of service over unreliable links is another challenging aspect of communication technology and different protocols and standards has been proposed for different applications. In mobile scenarios, location identification and handover process between different base stations must be addressed. Thus, network support for user mobility remains the topic of many ongoing research. On the other hand, service providers need to maintain a high level of connectivity and a good area coverage to deliver reliable network connections. This has led to the design of different types of networks including LAN. PAN, WAN, etc. And finally, while having all these features, like any other technology, wireless systems need to be cost efficient.

1.2 Centralized and Decentralized Networks

A major classification of wireless systems divides them into two big categories: centralized and decentralized networks (Figure 1.2). A centralized network is a type of network in which all of the nodes connect to a central server which is responsible of all the processing and controlling. In a centralized network, all the information flows through the server and nodes cannot talk to each other directly. In contrast, in a decentralized network, there is no server. Clients can communicate with each other freely and send and receive information directly. Control and processing tasks are distributed among the nodes and they cooperatively maintain the network. Ad hoc networks are the most interesting example of decentralized networks. In this dissertation nodes do not require to have global information about the network situation. They will operate and make decisions only according to the local information available to them. This enables us to build a decentralized configuration which eliminates the server requirement.

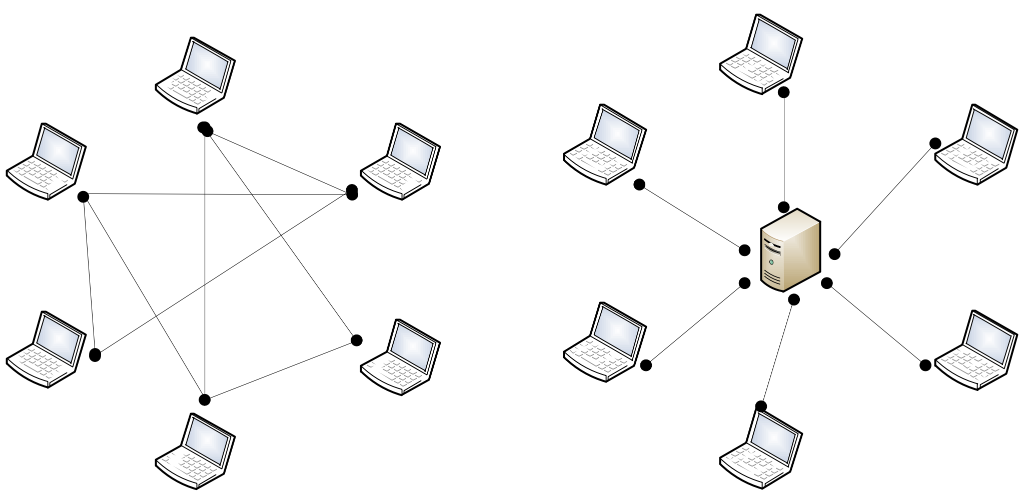

Figure 1.1: Left: A decentralized network and Right: A centralized network.

1.3 Wireless Ad-Hoc Networks

An ad-hoc network is a completely arbitrary combination of nodes that are communicating with each other. There is no fixed topology or backbone infrastructure in such a network. Each node, besides being a receiver or transmitter, acts as a router to relay information from a source to a destination. In an ad-hoc networks, nodes can enter and leave the network at any time and thus the topology of the network is completely dynamic. Nodes are responsible for keeping track of the network's connectivity to forward data to the designated destinations. This is in contrast to wired networks in which a set of routers is responsible for directing the data or to centralized networks where a base station manages communication among nodes. In other words transmission, access and routing strategies for these networks are generally ad-hoc. Ad-hoc networks provide a flexible network infrastructure for many emerging applications.

Chapter 2

Background

The problem of determining the capacity of wireless networks lies at the heart of network information theory. The main difficulties facing researchers to solve this problem are random behaviour of channel condition, interference of signals and other considerations like network topology and path loss. One other way to attack the problem, instead of trying to solve it, say, in an exact manner, is to look at it asymptotically. In other words, if we can not determine the behaviour of the network exactly, can we at least have a sense of that asymptotically when the number of nodes tends to infinity?

2.1 Asymptotic Behaviour of the Network

The pioneering work of Gupta and Kumar [1] was the first one to rigorously study the so-called capacity scaling laws for large wireless networks. Their focus was on the asymptotic throughput of the network when n, the number of nodes in the network tends to infinity. They show that in an arbitrary wireless network one can achieve a rate equal to $1/\sqrt{n}$ per node, and a rate equal to $1/\sqrt{n \log(n)}$ per node for a random network. Results of [1] where later improved by Franceschetti et. al. to show that a random network can also achieve a rate of $\sqrt{n}$ per node. Toumpis and Goldsmith [4] studied the same problem when delay and mobility constraints are imposed.

2.2 Single-Hop and Multi-Hop Network

Delivering information in a wireless network might be done via two main methods, single-hop and multi-hop. In a multi-hop network, data is routed to its designated destination via a number of intermediate nodes which are called routers or relays. The most notable advantage of multi-hop strategy is the high level of reliability that it provides. However there are also some shortcomings associated with this method. Routing data through intermediate nodes increases the delay in the network and thus might be inappropriate for delay sensitive application. On the other hand, the intrinsic complexity of such systems makes them more difficult to analyse–compared to single-hop systems. And more seriously, power consumption is higher in this type of networks. Relays consume power to receive, amplify and resend the received signal. Power inefficiency issue is more problematic in applications such as mesh networks where the access to individual nodes is limited or impossible, e.g. in a battle field. Such networks have been the topic of vast explorations in the recent years, see [5, 6].

The other method is single-hop communication in which despite the multi-hop networking, data is transmitted directly from the source to its designated destination and no other relay is supposed to convey the information to the receiver. In this dissertation we will be considering this type of wireless networks. See [11, 8]

2.3 Fading Channels

In many realistic scenarios, the transmitter and receiver are not in the line of sight of each other and most often there are many obstacles and scatters in the communication environment. Surfaces of tall buildings, cars and even earth and air can act as reflectors. Therefore, the transmitted signal may traverse many different paths to arrive its destination. At the destinations, these copies of the signal arrive with different delays, attenuations, and phase shifts. What the receiver receives, is actually a superposition of these signals. This in turn causes a distortion in the received signal which is called *fading*. We often exploit a stochastic processes (or random variable) to capture this non-deterministic behaviour of the *fading channel*. However, a simpler model may also be considered in which the only factor affecting the performance of communication is path loss attenuation, represented by a deterministic decaying power-law multiplier.

This simplified model is particularly appropriate in *extended networks*. An extended network is a network whose area increases as the number of its nodes n increases, thus the density remains constant. Therefore, large distances between nodes in such network is an important issue that might dominate other factors affecting the communication.

Contrary to extended networks, the area of a *dense network* is fixed and thus its density scales with n. In this case, the distance between nodes is small and power-law attenuation is not a reliable description of the channel. Therefore, for a more appropriate treatment of dense networks, one needs to consider fading as an important factor affecting the performance of the communication. Toumpis et al. show in [4] that fading can decrease the rate-per-node by a factor of $\log(n)$ when the same model as in [1] is assumed. The impact of fading is also studied extensively in multi-hop [4, 6, 9, 10] and single-hop [7, 8, 11, 12] networks. In [9] a constant transmission rate is assumed and to maximize the network throughput, the following strategy is proposed. Find disjoint paths from sources to destinations, such that they are constituted only from *good* enough links. Authors show that throughput of the network strongly depends on the fading distribution. In a later generalizations [10], a more complete model has been studied in which path loss attenuation is also taken into account.

In this work we first start in Chapter 3 with the basic model of fading channel where it is assumed that the only factor affecting the signal strength at the receiver is fading. Obviously, as discussed before, a more comprehensive model would also include path-loss. Such a model is considered in Chapter 4.

Chapter 3

Fading Channel Model

3.1 Network Model

We consider a wireless network consisting of n transmitters and n receivers located randomly in the plane. It is assumed that nodes are labelled such that transmitter i aims to send data to its corresponding receiver, receiver i. We also restrict ourselves to a special case of wireless networks where nodes are simple linear transmitters and receivers without multi-user detectors. In other words, source i can only talk to destination i in a fixed time slot and simultaneous sending and receiving is not possible. Communications occur in a single-hop manner, i.e., data would be sent from a source directly to its destination without relaying information through any other interior node. Therefore, there is a total of n communication links at each time slot. Associated with link i, there is a *direct channel gain* which represents the fading channel and will be denoted by h_{ii}, first index indicating the transmitter and second index representing the receiver. There are also *cross channel gains* h_{ji} which will be used to model the interference effect of source j at receiver i. The received signal $Y(i)$ at receiver i is therefore given by

$$Y(i) = h_{ii}X(i) + \sum_{j=1}^{n} h_{ji}X(j) + N(i)$$

where $X(i)$ represents the transmitted signal at transmitter i and $N(i) \sim \mathcal{N}(0, \sigma^2)$ is the background noise at receiver i. Assuming Gaussian signal transmission with constant power $Var(X(i)) = P$, the interference at each receiver can be treated as Additive White Gaussian Noise (AWGN) and hence the Shannon capacity formula [13] may be used to determine the achievable rate R_i at every link

$$R_i \leq \ln\left(1 + \frac{P|h_{ii}|^2}{\sigma^2 + P\sum_{j=1, j\neq i}^{n} |h_{ji}|^2}\right)$$

3.2 Problem Formulation

In delay-sensitive applications, each active link needs to support a minimum rate. Due to limited transmit power and interference caused by other active source-destination pairs, it is not always possible for all nodes to keep this minimum rate. Hence, only a subset of nodes having *good* channel conditions should be active while others remain silent during each time slot. An important question now arises about the maximum number of links that can be activated while supporting a minimum rate R_{min} and simultaneously not causing too much interference on the other active links in the network. An answer to this question will solve the famous rate-constrained throughput maximization problem studied extensively in the literature. Consider the communication network 3.1 and label the links with elements of $\mathbb{N}_n$, i.e. the set of natural numbers less than or equal to n. Also denote by $|\mathcal{A}|$ the cardinality of a set $\mathcal{A}$. The problem is to find a subset $\mathcal{A}^\star$ of the set of communication links such that $\mathcal{A}^\star$ has the maximal cardinality and all of the links in $\mathcal{A}^\star$ support the

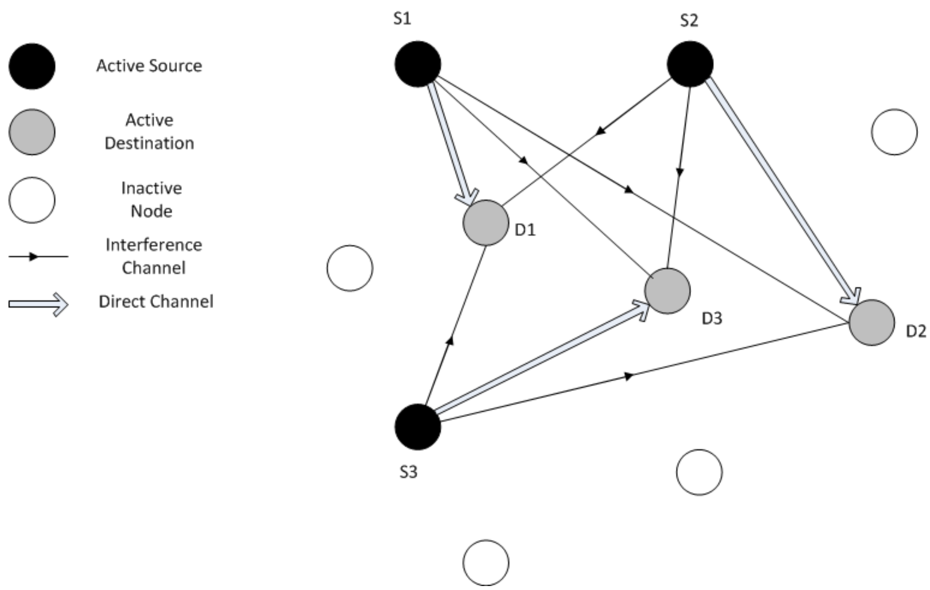

Figure 3.1: A communication network consisting of 5 transmitters and five receivers. Active sources and destinations are represented by black and white circles respectively, while white circles are silent nodes. Direct channels are thick arrows and interference channels are represented by thin mid-arrows.

minimum rate R_{min}. Since we are only interested in $|\mathcal{A}^\star|$, the problem can be translated as:

$$\begin{aligned} &\max\nolimits_{\mathcal{A}\subset\mathbb{N}_n} |\mathcal{A}| \\ &\text{s.t. } R_i > R_{min} \text{ for } i \in \mathcal{A} \end{aligned} \tag{3.1}$$

An answer to problem (3.1) implicitly conveys a random subset $\mathcal{A}^\star = \{i_1, i_2, \cdots i_{|\mathcal{A}^\star|}\}$ of the set of network links $\mathbb{N}_n$ such that for any link $i \in \mathcal{A}^\star$ the minimum rate can be maintained. Such a subset of links is illustrated in figure 3.1 by thick arrows. In this network, three out of five possible links are active. Obviously with fixed P and R_{min}, the set $\mathcal{A}^\star$ and it's cardinality completely depend on the channel gains h_{ij} $i, j \in \mathbb{N}_n$; and the interference caused by nodes transmitting at the same time. Thus $\mathcal{A}^\star$ is a random set depending on the specific realization of the network.

What we will show is that when channel gains are i.i.d. Rayleigh distributed, the asymptotic behaviour of the solution to (3.1) can be determined when the number of nodes in the network tends to infinity. In this chapter, we solve (3.1) for the case of ad-hoc networks where fading rather than path loss is important. This corresponds to the situation of dense networks where path loss between any pair of nodes is bounded and thus, we can focus on the fading gains. Next chapter is devoted to the case of extended networks where path loss attenuation plays a major role along with fading.

3.3 Main Result

Before we proceed to our problem, a review of some notation is necessary. If $f(x)$ and $g(x)$ are functions of one real variable x then
$f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
$f(x) = O(g(x))$ means $\lim_{x\to\infty} f(x)/g(x) < \infty$
$f(x) = o(g(x))$ means $\lim_{x\to\infty} f(x)/g(x) = 0$

Consider a wireless network with an arbitrary topology and assume that there are independent Rayleigh fading channels between different source-destination pairs; in other words, the channel gains $|h_{ij}|^2$ are independent realizations of the exponential distribution (See Appendix A). For such random variables one can easily show the following property:

Lemma 1. *For any exponential random variable X and constant $a > 0$,*

$$\mathbf{E}(X|X > a) = \mathbf{E}(X) + a$$
$$Var(X|X > a) = Var(X)$$

This lemma implies that for exponential random variables, conditioning does not change the variance and shifts the mean value, a consequence of the well known memoryless property. We now state the principal result of this chapter in the following theorem.

Theorem 2. *Under the assumption of independent Rayleigh fading channels for different source-destination pairs with channel gains $h_{ij} \sim \mathcal{CN}(0,1);\ i,j \in \mathbb{N}_n$, the typical behaviour of the maximum number of active links, M_n, determined by* (3.1)*, is governed by*

$$\lim_{n\to\infty} \mathbb{P}(\beta_1(n) \le M_n \le \beta_2(n)) = 1,$$

where

$$\beta_1(n) = \left(\frac{c_1 \ln(n)}{e^{R_{min}}}\right)^2 \tag{3.2}$$

$$\beta_2(n) = \left(\frac{c_2 \ln(n)}{e^{R_{min}}}\right)^2 \tag{3.3}$$

$$\tag{3.4}$$

and $0 < c_1 < 1$ and $1 < c_2 < \infty$ are arbitrary constants.

This theorem provides asymptotic bounds on the number of active links in the network. We now start by proving the lower bound.

Consider the wireless network 3.1 with independent channel gains $h_{ij} \sim \mathcal{CN}(0,1)$. As a result, $|h_{ij}|^2 \sim exp(1)$ is exponentially distributed with mean one[1]. As mentioned earlier, we have labelled the nodes in such a way that transmitter i's target for sending data is receiver i. For a fixed real number $h_0 > 0$, the direct link between transmitter i and receiver i is said to be a "good" link, if $|h_{ii}|^2 > h_0$. We exploit a link activation strategy which is based on the following criteria: a link will be active only if it is a "good" link. By

[1]In fact $|h_{ij}|^2 \sim exp(1/2)$, but for the sake of simplicity and without loss of generality we assume $|h_{ij}|^2 \sim exp(1)$. See Appendix A for details

choosing an appropriate value for h_0, one can show that every good link would be able to maintain the pre-set minimum rate.
Therefore our proof for the lower bound constitutes of two parts: in the first part we show that the number of good links in the network asymptotically almost surely (a.a.s.) is greater than $\beta_1(n)$; and in the second part it will be shown that all of such good links can support the minimum rate R_{min}. Putting these two results together reveals that there are at least a.a.s. $\beta_1(n)$ good links satisfying the minimum rate constraint.

Proof of the First Part: We show that there are a.a.s. $\beta_1(n)$ good links in the network having n links.

Let $p_0 = \mathbb{P}(|h_{ij}|^2 > h_0) = e^{-h_0}$ be the probability of a link being good. Now for $1 \leq i \leq n$ consider the Bernoulli sequence

$$\xi_i = \begin{cases} 1 & \text{if} \quad |h_{ii}|^2 \geq h_0 \\ 0 & \text{if} \quad |h_{ii}|^2 < h_0 \end{cases}$$

which assigns to the link i the value $\xi_i = 1$ if it is a good link and $\xi_i = 0$ otherwise. It is clear now that the number of good links has the same distribution as $M_n = \sum_{i=1}^{n} \xi_i$, which follows a Binomial distribution $B(n, p_0)$. Let $h_0 = m(n)^{\frac{1}{2}} e^{R_{min}}$ where $m(n)$ is an integer dependent on the size of the network n and having the property $e^{m(n)} \sim e^{\beta_1(n)}$. We will show that there is at least one set $\mathcal{A} \subset \mathbb{N}_n$ such that for every $i \in \mathcal{A}$, link i is a good link. Now for $m(n) \sim \beta_1(n)$,

$$\begin{aligned} p_0 = \exp(-h_0) &= \exp(-m(n)^{\frac{1}{2}} e^{R_{min}}) \sim \exp(-\beta_1(n)^{\frac{1}{2}} e^{R_{min}}) \\ &= \exp(-\frac{c_1 \ln(n)}{e^{R_{min}}} e^{R_{min}}) = \exp(-c_1 \ln(n)) = n^{-c_1} \end{aligned}$$

which gives $np_0 \sim n^{1-c_1}$. Having this result and noting that $m = o(n)$[2] one gets

$$\frac{(np_0 - m(n) + 1)^2}{2np_0} \sim \frac{(np_0)^2}{2np_0} = \frac{np_0}{2} \geq \frac{n^{1-c_1}}{2} \to \infty; \quad \text{as } n \to \infty \tag{3.5}$$

[2]since $m(n) \sim \beta_1(n) = O(\ln(n)^2)$

Moreover, we have asymptotically $m < np_0 + 1$ (in fact $m(n) = o(np_0)$), and the Chernoff bound[3][14] on the sum of independent random variables can be used as

$$\lim_{n\to\infty} \mathbb{P}(M_n \geq m) \geq \lim_{n\to\infty} \left(1 - \exp\left(-\frac{(np_0 - m + 1)^2}{2np_0}\right)\right) = 1 \tag{3.6}$$

which shows there are a.a.s. $m(n) \sim \beta_1(n)$ good links in the network. This completes the proof of the first part.

Proof of the Second part: Next, we need to show that every good link supports the minimum rate R_{min}. We will accomplish this task shortly, however, it is worth mentioning here that not only individual good links must be able to satisfy this constraint, but also the "set of all good links" must be able to maintain the minimum rate simultaneously. This concept will become more clear when we define stochastic rates.

Let $\mathcal{A}$, be a set of m so-called "good" links. For every real positive number ϵ (which will be precisely determined later according to our needs) and for every $i \in \mathcal{A}$ define the event $\mathcal{V}_{i,n}^{\epsilon}$ to be

$$\mathcal{V}_{i,n}^{\epsilon} = \left\{\omega \in \Omega : \left|\frac{1}{m-1} \sum_{j\in\mathcal{A}, j\neq i} (|h_{ji}|^2 \mathbf{1}_{[|h_{jj}|^2)>h_0]}) - \mathbb{E}(|h_{11}|^2)p_0\right| < \epsilon\right\} \tag{3.7}$$

note that the second term in the absolute value function above, is the expected value of every term in the summation, since for independent and identically distributed channel gains h_{ji} we have

$$\begin{aligned}\mathbb{E}(|h_{ji}|^2 \mathbf{1}_{(|h_{jj}|^2)>h_0)}) &= \mathbb{E}(|h_{ji}|^2)\mathbb{E}(\mathbf{1}_{(|h_{jj}|^2)>h_0)}) \\ &= \mathbb{E}(|h_{11}|^2)p_0; \quad \forall j, i \in \mathbb{N}_n,\, j \neq i\end{aligned}$$

Using Chernoffs's bound again,

$$\mathbb{P}\left(\Omega\backslash\mathcal{V}_{i,n}^{\epsilon}\right) = \mathbb{P}\left(\left|\frac{1}{m-1} \sum_{j\in\mathcal{A}, j\neq i} (|h_{ji}|^2 \mathbf{1}_{[|h_{jj}|^2>h_0]}) - \mathbb{E}(|h_{11}|^2)p_0\right| > \epsilon\right) \leq e^{-(m-1)I(\epsilon)}$$

[3] For a brief exposure to Chernoff bound see Appendix B

where $I(x)$ is the rate function defined as[4]

$$I(x) = \sup_{\theta>0}\{\theta x - \phi(\theta)\} \tag{3.8}$$

where $\phi(\theta) = \ln(M(\theta))$ and $M(\theta)$ is the moment generating function of the random variable $|h_{ji}|^2\mathbf{1}_{[|h_{jj}|^2>h_0]}$. We will show in lemma 3 that:

$$M(\theta) = \mathbb{E}(e^{\theta|h_{ji}|^2\mathbf{1}_{[|h_{jj}|^2>h_0]}}) = 1 + \frac{e^{-h_0}\theta}{1-\theta}, \quad \theta < 1$$

and therefore,

$$I(x) = (\sqrt{x} - \sqrt{p_0})^2$$

Lemma 3. *The rate function of the random variable* $|h_{ii}|^2\mathbf{1}_{[|h_{ji}|^2>h_0]}$ *is determined by*

$$I(x) = (\sqrt{x} - \sqrt{p_0})^2 \tag{3.9}$$

for x *satisfying* $p_0(1-\sqrt{p_0/x}) \ll \sqrt{p_0/x}$ *where* $p_0 = \mathbb{P}(|h_{ji}|^2 > h_0)$

Proof. Let us for simplicity put h_1 and h_2 for $|h_{ii}|^2$ and $|h_{ji}|^2$, respectively and define $H = h_1\mathbf{1}_{[h_2>h_0]}$. Therefore we have:

$$H = \begin{cases} h_1 & \text{if} \quad h_2 \geq h_0 \\ 0 & \text{if} \quad h_2 < h_0 \end{cases} \tag{3.10}$$

One can easily verify that the probability density function of the random variable H is $f_H(h) = (1-p_0)\delta(h) + p_0 f_{h_1}(h)$ where $\delta(h)$ is the Dirac delta function. Therefore, the moment generating function of the random variable H is derived as:

$$\begin{aligned} M(\theta) &= \mathbf{E}(e^{\theta|h_{ji}|^2\mathbf{1}_{[|h_{jj}|^2)>h_0]}}) = \mathbf{E}(e^{\theta H}) = \int_0^\infty e^{\theta h} f_H(h)dh \\ &= \int_0^\infty e^{\theta h}[(1-p_0)\delta(h) + p_0 f_{h_1}(h)]dh = 1 + \frac{p_0\theta}{1-\theta}, \quad \theta < 1. \end{aligned}$$

[4] See Appendix B

Now let us put $r(\theta) = \theta x - \ln(M(\theta))$. We know by definition that $I(x) = \sup_{\theta>0} r(\theta)$. One can approximate $r(\theta)$ with

$$\tilde{r} = \theta x - \left(\frac{p_0\theta}{1-\theta}\right) \tag{3.11}$$

if

$$p_0\theta \ll 1-\theta \tag{3.12}$$

To find the maximum of $r(\theta)$, we can alternatively find the maximum of $\tilde{r}(\theta)$ provided that the maximum occurs at a point where approximation (3.11) is valid, namely if condition (3.12) is met. Taking the first-order derivative of $\tilde{r}(\theta)$

$$\tilde{r}'(\theta) = x - \frac{p_0}{(1-\theta)^2} \tag{3.13}$$

and by solving $\tilde{r}'(\theta) = 0$ we have

$$\theta^\star = 1 - \sqrt{\frac{p_0}{x}} \tag{3.14}$$

We want to emphasis that, one can claim $\theta^\star$ maximizes $r(\theta)$ only when x is chosen in such a way that condition (3.12) is met[5]. Provided this, we finally have

$$I(x) = r(\theta^\star) \approx \hat{r}(\theta^\star) = \left(\sqrt{x} - \sqrt{p_0}\right)^2$$

as the lemma claims. $\square$

Now let us define the *stochastic rate* $X_{i,n}$ of link i in the network of n communication links by

$$X_{i,n} = \ln\left(1 + \frac{P|h_{ii}|^2\mathbf{1}_{[|h_{ii}|^2>h_0]}}{\sigma^2 + P\sum_{j\in\mathcal{A},j\neq i}|h_{ji}|^2\mathbf{1}_{[|h_{ji}|^2>h_0]}}\right) \tag{3.15}$$

The following lemma shows that any set of $\beta_1(n)$ "good" link, asymptotically almost surely supports the minimum rate R_{min}.

[5]For example, in the next part we need to choose $x = O(1/\ln(n))$. You can check that $\theta^\star = O(1 - \sqrt{p_0/\ln(n)^{-1}})$ satisfies condition 3.12

Lemma 4. *Every set of $\beta_1(n)$ good links can simultaneously support the minimum rate R_{min}.*

Proof. Let $\mathcal{A} \subset \mathbb{N}_n$ be a collection of good links satisfying $|\mathcal{A}| \sim \beta_1(n)$ and choose $\epsilon = O(\ln(n)^{-1}) = O(m^{-0.5})$. First note that as suggested by (3.8), on $\mathcal{V}_{i,n}^{\epsilon}$

$$\sum_{j\in\mathcal{A},j\neq i} (|h_{ij}|^2 \mathbf{1}_{[|h_{jj}|^2>h_0]}) - \mathbf{E}(|h_{11}|^2)p_0 \leq (m-1)(\mathbf{E}(|h_{11}|^2)p_0 + \epsilon) \tag{3.16}$$

$$= (m-1)(p_0+\epsilon) \tag{3.17}$$

where the equality follows from the fact that $|h_{11}|^2 \sim exp(1)$. Now for every $\omega \in \mathcal{V}_{i,n}^{\epsilon}$ and $i \in \mathcal{A}$, let us examining the stochastic rate of link i

$$\begin{aligned}
X_{i,n}(\omega) &= \ln\left(1 + \frac{P|h_{ii}|^2\mathbf{1}_{[|h_{ii}|^2>h_0]}}{\sigma^2 + P\sum_{j\in\mathcal{A},j\neq i}|h_{ji}|^2\mathbf{1}_{[|h_{ji}|^2>h_0]}}\right) \\
&\stackrel{a}{\geq} \ln\left(1 + \frac{Ph_0}{\sigma^2 + P(m-1)(p_0+\epsilon)}\right) \\
&= \ln\left(1 + \frac{Ph_0}{\sigma^2 + P(m-1)(e^{-h_0}+\epsilon)}\right) \\
&= \ln\left(1 + \frac{Pm^{0.5}e^{R_{min}}}{\sigma^2 + P(m-1)(e^{-m^{0.5}e^{R_{min}}}+\epsilon)}\right) \\
&\stackrel{b}{\sim} \ln\left(1 + \frac{Pm^{0.5}e^{R_{min}}}{P(m-1)m^{-0.5}}\right) \\
&\sim \ln\left(1 + e^{R_{min}}\right) \geq R_{min}
\end{aligned}$$

where a is true since by assumption link i is a "good" link and b is a consequence of the choice of ϵ. A direct consequence of this along with (3.8) is

$$\mathbb{P}(X_{i,n} < R_{min}) < \mathbb{P}(\Omega\backslash\mathcal{V}_{i,n}^{\epsilon}) < e^{-(m-1)I(\epsilon)} \tag{3.18}$$

Recalling (3), and our choice $\epsilon = O(\ln(n)^{-1})$ we have

$$I(\epsilon) = (\sqrt{\epsilon} - \sqrt{p_0})^2 = O\left(\sqrt{\frac{1}{\ln(n)}} - \sqrt{\frac{1}{n}}\right)^2 = O\left(\frac{1}{\ln(n)}\right)$$

therefore

$$\mathbb{P}(X_{i,n} < R_{min},\ \text{for some } i \in \mathcal{A}) \leq \sum_{i \in \mathcal{A}} \mathbb{P}(X_{i,n} < R_{min}) \tag{3.19}$$

$$\overset{a}{\sim} \ln(n)^2 \mathbb{P}(X_{i_0,n} < R_{min}, i_0 \in \mathcal{A}) \tag{3.20}$$

$$\overset{b}{\sim} \ln(n)^2 e^{-(m-1)I(\epsilon)} \tag{3.21}$$

$$\overset{c}{\sim} \ln(n)^2 e^{-(\ln(n)^2)/\ln(n)} \tag{3.22}$$

$$\sim \ln(n)^2/n \to 0 \tag{3.23}$$

where a is a consequence of $|\mathcal{A}| \sim \ln(n)^2$ and the fact that all $X_{i,n}$'s are identically distributed, b is true due to (3.18), and c is true due to the choice of ϵ. This completes the proof of the lemma and also the lower bound. □

Proof of the Upper bound. Now let us turn our attention to the upper bound and show that $M_n \leq \beta_2(n)$ holds asymptotically in probability, if a threshold-based link activation strategy is exploited. We need to show that

$$\lim_{n\to\infty} \mathbb{P}(M_n \leq \beta_2(n)) = 1 \tag{3.24}$$

or alternatively

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \geq \beta_2(n)\right) = 0 \tag{3.25}$$

Thus we only need to show that

$$\mathbb{P}(|h_{ii}|^2 \geq \beta_2(n)^{0.5} e^{R_{min}}) \to 0 \tag{3.26}$$

The probability that all links have channel gains less than h_0 equals $(1-p_0)^n$. As $h_0 = \beta_2(n)^{0.5} e^{R_{min}}$,

$$\mathbb{P}(|h_{ii}|^2 \geq \beta_2(n)^{0.5} e^{R_{min}};\ \text{for at least one } i) = 1 - (1-p_0)^n \tag{3.27}$$

which tends to zero if and only if

$$(1 - e^{-h_0})^n = (1 - e^{-h_0})^{e^{h_0} e^{-h_0} n} \to 1 \tag{3.28}$$

Since

$$(1 - e^{-h_0})^{e^{h_0}} \to e^{-1} \tag{3.29}$$

equation (3.28) holds if

$$\begin{aligned} ne^{-h_0} &= ne^{-\beta_2(n)^{0.5} e^{R_{min}}} = n \exp\left(-\frac{c_2 \ln(n)}{e^{R_{min}}} e^{R_{min}}\right) \\ &= ne^{-c_2 \ln(n)} = n^{1-c_2} \to 0 \end{aligned}$$

which holds as $1 < c_2 < \infty$ by assumption.

Now, we only need to show that $X_{i,n}$ is a.a.s. zero on $\mathcal{V}_{i,n}^{\epsilon}$ as $n \to \infty$. As channel gains are exponentially distributed, we have

$$\begin{aligned} \mathbf{E}(X_{i,n}|(\mathcal{V}_{i,n}^{\epsilon})^c) &= \mathbf{E}\left(\ln\left(1 + \frac{P|h_{ii}|^2 \mathbf{1}_{[|h_{ii}|^2 \geq h_0]}}{\sigma^2 + \sum_{j \in \mathcal{A}, j \neq i} P|h_{ji}|^2 \mathbf{1}_{[|h_{jj}|^2 \geq h_0]}}\right) |(\mathcal{V}_{i,n}^{\epsilon})^c\right) \\ &\overset{a}{\leq} \ln \mathbf{E}(1 + \frac{P}{\sigma^2}|h_{ii}|^2 \mathbf{1}_{[|h_{ii}|^2 \geq h_0]}) \\ &= \ln(1 + \frac{P}{\sigma^2}\mathbf{E}(|h_{ii}|^2 \mathbf{1}_{[|h_{ii}|^2 \geq h_0]})) \\ &= \ln(1 + \frac{P}{\sigma^2}\mathbf{E}(|h_{ii}|^2 \mid |h_{ii}|^2 \geq h_0)) \\ &\overset{b}{=} \ln(1 + \frac{P}{\sigma^2}\mathbf{E}(|h_{ii}|^2 + h_0)) \\ &= \ln(1 + \frac{P}{\sigma^2}(1 + h_0)) \end{aligned}$$

where a is based on Jenson's inequality and follows lemma 1.

Consequently, as the number of active links goes to infinity

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}(X_{i,n}|\mathcal{V}_{i,n}^{\epsilon})^c) &\leq \ln(\frac{Ph_0}{\sigma^2}) \\
&\leq \ln\left(\frac{P\beta_2(n)^{0.5}e^{R_{min}}}{\sigma^2}\right) \\
&= \ln\left(\frac{Pc_2}{\sigma^2}\ln(n)\right)
\end{aligned}
$$

Hence, using 3.8 and choosing again $\epsilon \sim 1/\ln(n)$, as $n \to \infty$ we have:

$$
\begin{aligned}
\mathbb{E}(X_{i,n}\mathbf{1}_{(\mathcal{V}_{i,n}^{\epsilon})^c}) &= \mathbb{E}(X_{i,n}|(\mathcal{V}_{i,n}^{\epsilon})^c)\mathbb{P}((\mathcal{V}_{i,n}^{\epsilon})^c) \\
&\leq \ln\left(\frac{Pc_2}{\sigma^2}\ln(n)\right)e^{-c_2\ln(n)} \\
&= \frac{1}{n^{c_2}}\ln\left(\frac{Pc_2}{\sigma^2}\ln(n)\right) \to 0
\end{aligned}
$$

This completes the proof. □

3.3.1 Discussion

Our choice for the threshold $h_0 = c_1 \ln(n)$ follows directly the fact that the maximum of n i.i.d. exponential random variables behaves asymptotically almost surely as $ln(n)$. In mathematical words, if $Z_n = \max_{1\leq i\leq n}(X_1, X_2, \cdots X_n)$ then

$$\lim_{n\to\infty} \frac{Z_n}{\ln(n)} = 1, \quad a.s.$$

A direct conclusion of this is

$$\mathbb{P}\left(Z_n > c\ln(n)\right) = 1, \text{ as } n \to \infty$$

for $c \leq 1$ and

$$\mathbb{P}\left(Z_n > c\ln(n)\right) = 0, \text{ as } n \to \infty$$

for $c > 1$. This last equality makes it clear why any threshold $C_2 \ln(n)$ with the constant coefficient $C_2 > 1$ is an inappropriate link-activation threshold; namely, for such a threshold, non of the channel gains can satisfy $h_{ii} > h_0$ and thus no link would be activated.

Chapter 4

Path Loss Attenuation Model

4.1 Network Model

In this chapter we build on top of the purely-fading network model previously studied in Chapter 3 and add to it path loss attenuation caused due to the large distances between nodes in extended networks. We model path loss by a simple attenuation factor D_{ij}^{α}, where D_{ij} denotes the Euclidiean random distance between source i and destination j, and α represents the path loss exponent. We also assume that these distances are independent exponential random variables having mean λ. To model the multi-path effect, independent small-scale fading channels between different source-destination pairs are considered. We assume multi-path fading gains X_{ij} are independent realizations of an exponential distribution having mean one. Accounting for both multi-path and path-los, the channel between transmitter i and receiver j is characterized by

$$h_{ij} = X_{ij} D_{ij}^{-\alpha} \tag{4.1}$$

4.2 Tail Behaviour of the Channel Gains

Let h, X and D be random variables having the same distribution as h_{ij}, X_{ij} and D_{ij} respectively. We will show that

$$\mathbb{P}\left(h > z\right) \sim \frac{c_1}{z^{1/\alpha}} \tag{4.2}$$

for some constant c_1 dependent only on the network parameters α and λ and to be determined shortly. The strategy is to show a tight upper bound and a lower bound for the probability $\mathbb{P}\{h > z\}$ which converges to the right hand side of (4.2). We begin by the upper bound here.

$$\mathbb{P}\left(h > z\right) = \mathbb{P}\left(XD^{-\alpha} > z\right) = \mathbb{P}\left(D^{\alpha} z < X\right) = \int_0^{\infty} \mathbb{P}\left(D^{\alpha} z < X | X = x\right) f_X(x) dx$$

$$= \int_0^{\infty} \mathbb{P}\left(D < (\frac{x}{z})^{\frac{1}{\alpha}}\right) e^{-x} dx = \int_0^{\infty} (1 - e^{-\lambda(\frac{x}{z})^{\frac{1}{\alpha}}}) e^{-x} dx = 1 - \int_0^{\infty} e^{-[\lambda(\frac{x}{z})^{\frac{1}{\alpha}} + x]} dx$$

Now let us consider the last integral more precisely. Using the integration by parts formula, and letting $c = \lambda / z^{1/\alpha}$ and $\beta = 1/\alpha$ we have:

$$\int e^{-[\lambda(\frac{x}{z})^{\frac{1}{\alpha}} + x]} dx = \int e^{-cx^{\beta}} e^{-x} dx = \int e^{-cx^{\beta}} d(-e^{-x}) = -e^{-x} e^{-cx^{\beta}} + \int e^{-x} d(e^{-cx^{\beta}})$$
$$= -e^{-x} e^{-cx^{\beta}} - c\beta \int e^{-cx^{\beta}} e^{-x} x^{\beta - 1} dx$$

therefore:

$$\int_0^{\infty} e^{-[\lambda(\frac{x}{z})^{\frac{1}{\alpha}} + x]} dx = 1 - c\beta \int_0^{\infty} e^{-cx^{\beta}} e^{-x} x^{\beta - 1} dx$$

which results in

$$\mathbb{P}\left(h > z\right) = c\beta \int_0^{\infty} e^{-cx^{\beta}} e^{-x} x^{\beta - 1} dx = c\beta I(c)$$

where $I(c) = \int\limits_0^\infty e^{-cx^\beta} e^{-x} x^{\beta-1} dx$. We know (by the definition of Gamma function) that

$$\int\limits_0^\infty e^{-x} x^{\beta-1} dx = \Gamma(\beta)$$

and since $c > 0$,

$$I(c) = \int\limits_0^\infty e^{-cx^\beta} e^{-x} x^{\beta-1} dx \leq \Gamma(\beta)$$

thus

$$\mathbb{P}(h > z) = c\beta I(c) \leq c\beta\Gamma(\beta) = \frac{\Gamma(1/\alpha)\lambda}{\alpha z^{\frac{1}{\alpha}}}$$

Having established an upper bound, we will now prove the lower bound $\mathbb{P}(h > z) \geq \frac{\Gamma(1/\alpha)}{\alpha}(\frac{\lambda}{z^{1/\alpha}})^{1+\delta}$ for any real constant $\delta > 0$[1]. Let us intuitively examine the behaviour of the integral $I(c) = \int\limits_0^\infty e^{-cx^\beta} e^{-x} x^{\beta-1} dx$ as $c \to 0$ or equivalently as $z \to \infty$ (recall that $c = \lambda/z^{1/\alpha}$). We will consider this integral as a function of c and compare it to $c^\delta\Gamma(\beta)$ for any given value of $\delta > 0$. Note that:

$$\lim_{c\to 0} I(c) = \Gamma(\beta)$$

and hence

$$\lim_{c\to 0} c^\delta\Gamma(\beta) = 0, \quad \text{for} \quad \delta > 0$$

Therefore, for small enough c, or large enough z, we will have $I(c) > c^\delta\Gamma(\beta)$. Substituting this result into (13) gives:

$$\mathbb{P}(h > z) = c\beta I(c) > c^{1+\delta}\beta\Gamma(\beta) = \frac{\Gamma(1/\alpha)}{\alpha}(\frac{\lambda}{z^{1/\alpha}})^{1+\delta}$$

[1] note that this would be a lower bound for all values of δ only when z is big enough

for large enough z –not even necessarily $z \to \infty$. Noting that this result is true for any constant $\delta > 0$, and by combining the lower and upper bounds, we get

$$\begin{aligned}\mathbb{P}\left(h > z\right) &= \frac{\Gamma(1/\alpha)\lambda}{\alpha z^{1/\alpha}}, \text{ for large enough } z \\ &= \frac{c_1}{z^{1/\alpha}} \end{aligned} \tag{4.3}$$

4.3 Main Result

We will now prove some asymptotic results on M_n, the largest number of active links. Here we follow the same link activation strategy as the previous chapter, namely, the link between transmitter i and receiver i would be active only if $h_{ij} > z_0$ for some real-valued threshold $z_0 > 0$ to be determined later. We show that the number of active links M_n is asymptotically $n^{1-\gamma}$ for some $\gamma < 1$.

As before, define Bernoulli random variables ξ_i:

$$\xi_i = \begin{cases} 1 & \text{if} \quad h_{ii} \geq z_0 \\ 0 & \text{if} \quad h_{ii} < z_0 \end{cases} \tag{4.4}$$

then the number of activated links has the same distribution as $M_n = \sum_{i=1}^{n} \xi_i$ which satisfies a Binomial distribution $B(n, p_0)$ where n is the total number of communication links and $p_0 = \mathbb{P}(h > z_0)$ is the probability of a link being active. Let $z_0 = c_1^{1/\beta} m(n)^x$ for some integer value $m(n)$ dependent on the network size n, real constant $x < 1$ and c_1 as obtained earlier in (4.3). The integer number m can be interpreted as the possible number of active links (and not necessarily the maximum). If p_0 is the probability of a link being activate, then we have:

$$p_0 = \mathbb{P}\left(h_{ii} > z_0\right) = c_1 z_0^{-\beta} \sim c_1 (c_1^{1/\beta} m^x)^{-\beta} = m^{-\beta x}$$

Now let $m < n^{\gamma/\beta x}$ (i) for some constatnt $0 < \gamma < 1$. According to (4.5) this results in $p_0 > n^{-\gamma}$ and hence $np_0 > n^{1-\gamma}$ (ii). If we further require that $x > \gamma/\beta(1-\gamma)$ (iii) then one can see $m < np_0$:

$$\begin{aligned} np_0 &\overset{a}{\sim} nm^{-\beta x} \\ &\overset{b}{>} m^{\beta x/\gamma} m^{-\beta x} \\ &= m^{\beta x \frac{1-\gamma}{\gamma}} \\ &\overset{c}{>} m \end{aligned}$$

where a comes from 4.5, b from (i), and c from (iii). Having this result, we can use the Chernoff bound on the sum of independent Bernoulli trials as

$$\begin{aligned} \mathbb{P}\left(M_n \geq m\right) &\geq 1 - \exp\left(-\frac{1}{2p_0}\frac{(np_0 - m + 1)^2}{n}\right) \\ &\sim 1 - \exp\left(-\frac{np_0}{2}\right) \\ &\to 1 \quad \text{as} \quad n \to \infty \end{aligned}$$

according to (ii) and $\gamma < 1$. Combining (i) and (iii) also results in

$$m < n^{1-\gamma} \tag{4.5}$$

which shows that the number of active links can get arbitrarily close to n and is in fact of the order of n. Thus, we proved that as n goes to infinity, we probability approaching one, there are at least $M_n = n^{1-\gamma}$ link channels with $h_{ii} > z_0$.

Next we show that the minimum rate constraint is satisfied asymptotically almost surely for a set of m channel gains. Suppose as before that $\mathcal{A} \subset \mathbb{N}_n$ is a collection of m "good" channel links and define new random variables $g_{ij} = h_{ij}1_{[h_{ij}>\epsilon_0]}$, $1 \leq i, j \leq n$ for some small $\epsilon_0 > 0$. This means that g_{ij} is essentially the same as h_{ij} whenever $h_{ij} > \epsilon_0$. More formally, for $z_0 > \epsilon_0$ we have:

$$\mathbb{P}(g_{ij} \geq z_0) = \mathbb{P}(h_{ij} 1_{[h_{ij} > \epsilon_0]} \geq z_0) \tag{4.6}$$

$$= \mathbb{P}(h_{ij} \geq z_0) \tag{4.7}$$

the reason for this definition will become clear later in the proof where we need the random variables to have finite expectation.

Let g be identically distributed as g_{ij}. We show that g^p have finite expectation for $p = y/\alpha$ where $y < 1$ is some real number. The tail behaviour is:

$$\begin{aligned}\mathbb{P}\left(g^p > z_0\right) &= \mathbb{P}(g > (z_0)^{1/p}) \\ &= \frac{c_1}{(z_0^{\alpha/y})^{\frac{1}{\alpha}}} \\ &= \frac{c_1}{z_0^{1/y}}\end{aligned}$$

since $1/y > 1$ and also g is zero in an ϵ-neighbourhood of the origin, the expected value of g^p is finite.

Now we can use the following version of the law of large numbers [15, Theorem. 2.1.5]

Theorem 5. *suppose that $p \in (0, 2)$ and $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. random variables. The strong law of larger numbers:*

$$n^{-1/p}(S_n - an) \to 0, \ \ a.s. \tag{4.8}$$

holds for some real constant a if and only if $E|X|^p < \infty$. If $\{X_n\}$ obeys the SLLN then we can choose

$$a = 0 \qquad if \qquad p < 1 \tag{4.9}$$

Now let us define

$$\mathcal{U}_{i,n}^{\epsilon} = \left\{ \omega \in \Omega : \left| \frac{1}{(m-1)^p} \sum_{j \in \mathcal{A}, j \neq i} h_{ji} 1_{\{h_{jj} \geq z_0\}} \right| < \epsilon \right\}$$
$$= \left\{ \omega \in \Omega : \left| \frac{1}{(m-1)^p} \sum_{j \in \mathcal{A}, j \neq i} g_{ji} \right| < \epsilon \right\}$$

According to theorem (5) and noting that in our case $p = y/\alpha < 1$, we have $\mathbb{P}(\mathcal{U}_{i,n}^{\epsilon}) \to 1$ as $n \to \infty$.

Now let us examine the stochastic rates previously defined in (3.15):

$$X_i = ln \left(1 + \frac{P g_{ii}}{\sigma^2 + \sum_{j \in \mathcal{A}, j \neq i} P g_{ji}} \right)$$

Choose $\epsilon = O(c^{1/\beta} e^{-R_{min}} m^{x-p})$. For channel gains satisfying $g_{ii} > z_0$ on $\mathcal{U}_{i,n}^{\epsilon}$ we have:

$$\begin{aligned}
\ln \left(1 + \frac{P g_{ii}}{\sigma^2 + \sum_{j \in \mathcal{A}, j \neq i} P g_{ji}} \right) &\geq \ln \left(1 + \frac{P z_0}{\sigma^2 + P(m-1)^p \epsilon} \right) \\
&\sim \ln \left(1 + \frac{z_0}{(m-1)^p \epsilon} \right) \\
&\sim \ln \left(1 + \frac{c^{1/\beta} m^x}{(m-1)^p c^{1/\beta} e^{-R_{min}} m^{x-p}} \right) \\
&\sim \ln(1 + e^{R_{min}}) \\
&\geq R_{min}
\end{aligned}$$

This affirms that as $n \to \infty$, with probability approaching one, all of the links support the minimum rate and thus can be activated. Note that choice of $\epsilon = O(c^{1/\beta} e^{-R_{min}} m^{x-p})$ imposes another condition on the variables involved, i.e., we must have $x - p < 0$ or equivalently, $y > \alpha x$ [2].

[2] remember that $p = y/\alpha$

4.4 A Discussion About the Result

So far we have shown that

$$\mathbb{P}\left(\lim_{n\to\infty} X_{i,n} > R_{min}\right) = 1, \quad \forall i \in \mathcal{A}_n \tag{4.10}$$

the above "almost sure" equality needs to be explained further for its power to be fully understood. For every $n \in \mathbb{N}$, we have shown that there is a set $\mathcal{A}_n$ of "good" active links such that $|\mathcal{A}_n| \sim n$ and every link $i \in \mathcal{A}_n$ supports the minimum rate R_{min} asymptotically almost surely. Now let $\{i_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that $i_n \in \mathcal{A}_n$, $\forall n \in \mathbb{N}$. Equation (4.10) can be interpreted as the following equivalent equality

$$\mathbb{P}\left(\lim_{n\to\infty} X_{i_n,n} > R_{min}\right) = 1 \tag{4.11}$$

Since $|\mathcal{A}_n| \sim n$, we have

$$\mathbb{P}\left(i \in \limsup_{n\to\infty} \mathcal{A}_n\right) = 1, \quad \forall i \in \mathbb{N} \tag{4.12}$$

where

$$\limsup_{n\to\infty} \mathcal{A}_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathcal{A}_n$$

This affirms that for any integer i there is a sequence $\{i_n\} \in \mathbb{N}^{\mathbb{N}}$ such that $i_n = i$, infinitely often. Now, if we write (4.11) for such a sequence (which is infinitely often equal to constant i), and note that the limit of the constant subsequence is the same as that of the sequence itself, one can infer

$$\mathbb{P}\left(\lim_{n\to\infty} X_{i,n} > R_{min}\right) = 1, \quad \forall i \in \mathbb{N} \tag{4.13}$$

This last equality is very similar to that of equation (4.10), but they differ in the sense that the latter is true for every constant link $i \in \mathbb{N}$ whereas the former is only true for not-predetermined links in the set of active links $\mathcal{A}_n$.

A physical interpretation of (4.13) is this: consider a network of two communication nodes, constituting one link which is active while supporting R_{min}. Now, add another pair of nodes which are supposed to communicate which each other. The interference caused by their communication may cause links 1 to shut down, or vice versa, we might keep the newly added link silent in order for the first link to maintain quality; or maybe both of the links can be active while supporting R_{min}. Continue this process and in every step, add a new pair of nodes that want to talk to each other and cause interference on the other communications if they do so. Equation (4.13) now says that asymptotically almost surely, link 1 can remain active while supporting the minimum rate. This is also true for any other link: suppose you have a network of n communication links in which, say, link i is active. If we keep track of the statues of this fixed link while adding other pairs, we will see that this links can remain active as the network gets larger, almost surely.

Now, as in the case of simple fading model without path-loss attenuation, we also need to show that not only individual links in $\mathcal{A}_n$ can support the minimum rate, but also can do the whole set of $O(n)$ active links.

Define

$$H_i = \left\{\omega | \lim_{n\to\infty} X_{i,n} > R_{min}\right\}, \quad \forall i \in \mathbb{N} \tag{4.14}$$

since $\mathbb{P}(H_i) = 1$ for all $i \in \mathbb{N}$, we have

$$\mathbb{P}\left(\bigcap_{i\in\mathbb{N}} H_i\right) = 1 \tag{4.15}$$

this completes the proof.

In this chapter we showed that once the path loss attenuation between nodes is taken into account, the number of active links dramatically increases compared to the pure fading model. This result is also intuitive in the sense that in an extended network, where nodes are far distant from each other, one expects the interference caused by other communications to be weaker compared to a dense network with the same number of links. This decrease in the interference increases the signal-to-noise-plus-interference-ratio and hence, more links are qualified to be active.

Chapter 5

Conclusion

In this dissertation we solved the rate-constrained throughput maximization problem for two random networks. Our approach is to find the maximum number of communication links that can support a minimum rate R_{min}. Both network models consist of $2n$ nodes, n of them transmitters and n of them receivers. Every transmitter aims to send data to its designated destination in a single-hop manner. There is also no broadcast nor multi-cast embedded in the network, thus there is a total of n communication links. We propose a threshold-based link activation strategy to gain a lower bound on the greatest number of active links. What this means is that, a link would be active if its channel gain exceeds an appropriately chosen threshold, otherwise it would be silent. Links having this threshold property, are called *good* links. In chapter 3, we solved the problem for a pure fading network model, i.e. a network in which the prominent factor affecting the quality of the communications is the fading nature of the channels, as well as interference received from other communications. This model arises in situations where a high number of nodes are located in a confined area, and distances between nodes are small. By using Chernoff bound on the sum of independent Bernoulli random variables, we first show that asymptotically almost surely (a.a.s.) there exists a set of $O(\ln(n)^2)$ good links in the network, and then that all of the links in this set can simultaneously support the rate R_{min}. Chapter 4 was devoted to a more general case where we put on top of the fading model path loss attenuation coefficients which are exponential random variables. Here we prove that a.a.s. all of the n links can be active while supporting R_{min}. Here, a channel gain between transmitter i and receiver j is determined by $h_{ij} = X_{ij}D_{ij}^{-\alpha}$ where $h \sim Exp(1)$ represents the fading effect and $D_{ij} \sim Exp(\lambda)$ represents the Euclidean distance between transmitter and receiver. First we show that channel gains h_{ij} have Pareto-like tails. The proof of the asymptotic bound for this model, again consists of two parts. In the first part we show that there are at least $O(n^{1-\gamma})$ good links. Then we use strong law of large numbers to show that all of these links can be activated while supporting R_{min}. Our choice for the exponential distribution of the path loss attenuation factor D_{ij} is motivated by the fact that in a network where nodes are distributed according to the two dimensional Poisson distribution, the distance between a node and its nearest neighbour is an exponential random variable. Whereas this is not the case for the distance between that node and its second or other nearest neighbours. Therefore, one can see that different gains contributing in the interference term, in the denominator of the stochastic rate, are no longer identically distributed, since, they are associated for example to the second, seventh and tenth nearest neighbours and associated D_{ij}'s have different distributions. Solving the problem for this general case remains an open problem and topic for a future research.

APPENDICES

Appendix A

Rayleigh and Exponential Random Variable

A Rayleigh random variable R can be characterized as the magnitude of a complex normal random variable, which itself has two i.i.d. Gaussian random variables as real and imaginary parts. In a more formal language, suppose X and Y are two independent and identically distributed normal random variables with mean zero and variance σ. Then $Z = X + iY$ is a complex normal random variable and $R = |Z| = \sqrt{X^2 + Y^2}$ is Rayleigh distributed with parameter σ. The probability distribution function of the Rayleigh random variable R is

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0 \tag{A.1}$$

and its cdf is

$$F_R(r) = 1 - e^{-r^2/2\sigma^2} \tag{A.2}$$

From this definition, it is also clear that if $R \sim \mathcal{R}$, then R^2 which is the sum of two i.i.d. standard normal random variables, has a chi-squared distribution with two degrees of freedom.

Now let us derive another distribution for R^2. From elementary probability we know that if $g(\cdot)$ is a non-decreasing mapping on the real line, and if $F_X(x)$ is the cdf of random variable X, then $F_X(g^{-1}(y))$ is the cdf of the new random variable $Y = g(X)$. Using this fact and putting $R^2 = S$, we have

$$F_S(s) = 1 - e^{s/2\sigma^2} \tag{A.3}$$

which is the cdf of an exponential random variable with parameter $1/2\sigma^2$. Thus we have shown that if $R \sim \mathcal{R}(\sigma)$ then $R^2 \sim Exp(1/2\sigma^2)$.

Appendix B

Chernoff Bound for the Sum of Random Variables

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of real-valued independent and identically distributed random variables with the same distribution as X. The moment generating function is defined as

$$M_X(h) = \mathbb{E}(e^{hX_1}) \tag{B.1}$$

and let us assume that there exists h^* such that $M_X(h) < \infty$ if $|h| < h^*$. using the inequality $|x| \leq h^{-1}(e^{hx} + e^{-hx})$, one can show that this finiteness condition is sufficient for the first moment to exist. Now let us denote by m and S_n the mean value $\mathbb{E}(X_1)$ and the partial sum $\sum_{i=1}^{n} X_i$ respectively. Then for any real $a > m$ we have

$$\mathbb{P}\left(\frac{S_n}{n} > a\right) \leq e^{-nI_X(a)} \tag{B.2}$$

where $I_X(a)$ is called the rate function and defined to be

$$I_X(a) = \sup_{h>0}\{ah - \phi(h)\} \tag{B.3}$$

where $\phi_X(h) = \log(M_X(h))$ is the logarithmic moment generating function.

proof we first show an special case for $n = 1$ random variable and then extend the result for the more general case. Let $\mathbb{E}(X) = m$ and $a > m$ be a real number. Then

$$\mathbf{1}_{[X>a]} \leq e^{h(X-a)} \ ; \ \ \forall h > 0 \tag{B.4}$$

taking expectation from both sides,

$$\mathbb{P}(X > a) < \mathbb{E}(e^{h(X-a)}) \tag{B.5}$$

$$= e^{-(ha-\phi_X(h))} \tag{B.6}$$

since this inequality is valid for any $h > 0$ we can seek to minimize the right hand side to get a tight bound, or equivalently to maximize the exponent $ha - \phi_X(h)$ which by definition is the rate function $I_X(a)$. This result can be simply extended to the more general case of n i.i.d. random variables. First, note that if $X_1, X_2, \cdots, X_n$ are n i.i.d. random variables having the common moment generating function $M_X(h)$, then the moment generating function of the random variable $S_n = \sum_{i=1}^{n} X_i$ would be $M_X(h)^n$. knowing this and writhing the Chernoff bound for S_n we have

$$\mathbb{P}(S_n \geq na) \leq e^{-I_{S_n}(na)} \tag{B.7}$$

where

$$I_{S_n}(na) = \sup_{h>0}\{nah - \phi_{S_n}(h)\} \tag{B.8}$$

$$= \sup_{h>0}\{nah - n\phi_X(h)\} \tag{B.9}$$

$$= n \sup_{h>0}\{ah - \phi_X(h)\} \tag{B.10}$$

$$= nI_X(a) \tag{B.11}$$

and this completes the proof

Appendix C

Heavy-Tailed Distributions and the "Single Big Jump" Property

Heavy-tailed distributions arise frequently in the analysis of many stochastic systems. They have been proved to be useful in modelling communication networks, financial markets, physical systems and economical phenomena. Some of the most notable heavy-tailed distributions are Pareto, lognormal and Weibull distributions. In fact many of the commonly used heavy-tailed distribution fall inside one of these categories. We will shortly define these distributions.

the *tail function* $\bar{F}_X$ of a distribution F_X on real line $\mathbb{R}$ is defined by $\bar{F}_X(x) = 1 - F_X(x) = \mathbb{P}(X > x)$. We will occasionally omit the index X referring to the random variable and simply write $\bar{F}$ instead of F_X. A *tail property* of the distribution F is then defined to be a property which can be characterized merely by $\bar{F}$. The distribution F is also said to have *right-unbounded support* if $\bar{F}(x) > 0$ for all x. The formal definition of a heavy-tailed distribution is given below.

Definition 1. *A distribution F is said to be heavy-tailed if*

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) = \infty, \quad \forall \lambda > 0$$

The following theorem shows heavy-tailedness is a actually a tail property. To connect heavy-tailedness property of a distribution to the same property of a function, we need the following definition.

Definition 2. *a function f is said to be heavy-tailed if*

$$\limsup_{x\to\infty} f(x) e^{\lambda x} = \infty, \; \forall \lambda > 0$$

Now it is clear that heavy-tailedness is a tail property of a function. The following theorem is of special interest to us

Theorem 6. *for any distribution F the following statements are equivalent:*
1- F is a heavy-tailed distribution.
2- the function $\bar{F}$ is heavy-tailed.

C.1 Some Examples of Heavy-Tailed Distributions

To get a better idea of how heavy-tailed distributions behave, let us consider some of the most popular examples of them.

- The *Pareto distribution* is the most classical example. The tail function $\bar{F}$ of a Pareto random variable is given by
$$\bar{F}(x) = \left(\frac{c}{x+c}\right)^r$$
where c and r are real positive constants. It is obvious that $\bar{F}(x) \sim (c/x)^r$ as $x \to \infty$ and this explains why Pareto distribution is sometimes called as *power law distribution*. All moments of order less than r are finite and moments of order greater than or equal to r are infinite.

- The Burr distribution has tail function $\bar{F}$ given by

$$\bar{F}(x) = \left(\frac{c}{x^\beta + c}\right)^r$$

for some parameters $r, \beta, c > 0$. The Burr distribution is very similar to th ePareto distribution and it is in fact a slight generalization of the Pareto distribution. All moments of the order $k < \beta r$ are finite and those of the order $k \geq \beta r$ are infinite.

- The *Cauchy distribution*. Its probability density function is given by

$$f(x) = \frac{c}{\pi((x-a)^2 + c^2)}$$

where $c > 0$ and $a \in \mathbb{R}$ are called the *scale* and *position* parameter. In this case, moments of order $k < 1$ are finite and those of order $k \geq 1$ are infinite.

- The *log-normal distribution* is another example of heavy-tailed distributions. A log-normal distribution wit parameters μ and $\sigma > 0$ is actually the distribution of a random variable whose logarithm is normally distributed with mean μ and variance σ^2. The pdf of such a random variable is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right)$$

All moments of the lognormal distribution are finite.

- The *Weibull distribution* has the tail function $\bar{F}$ given by

$$\bar{F}(x) = e^{-(x/c)^\alpha}$$

where the scale parameter $c > 0$ and shape parameter $\alpha > 0$ are given real constants. The Weibull distribution would be heavy-tailed if $\alpha < 1$, otherwise it is a light-tailed distribution. For $\alpha = 1$, the Weibull distribution is actually the exponential distribution. All moments of the Weibull distribution are finite.

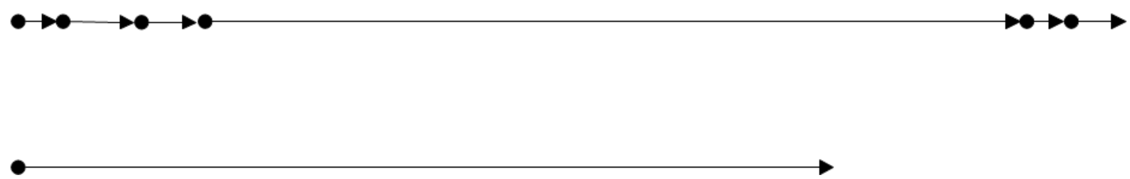

Figure C.1: If the step sizes are drawn from a heavy-tailed distribution on the real line, then the biggest step is almost as good as the sum of them.

C.2 Subexponential Distribution and "Single Big Jump"

The important property of the heavy-tailed distributions that we want to emphasize here, is the so called "single big jump phenomena". This property can be best described through an example. Suppose that you are playing a game where you begin from a starting pint and are allowed to take ten steps each at a time. Step sizes are positive real numbers drawn from a given distribution and independent from each other. Your goal is to reach as far a distance as possible. Now suppose you have two choices:

1. to start your next step from where you finished in the previous step, or
2. to return to your start point at every step, and at the end, choose the largest step that you have had.

It is obvious that the first strategy leads always to better results since the maximum step is among the ten steps, and all of the step sizes are positive–there are no backward movements. But there is an exception. If the step sizes are drawn from a heavy-tailed distribution on positive real numbers, say for example, from a Pareto or lognormal distribution, the second strategy described above works almost as well as the first one–See figure C.2. In other words, your biggest step will displace you almost as far as the sum of the ten steps taken. This concept, put into mathematical notation is as follows. Let

$X_1, X_2, \cdots X_n$ be some random variables with heavy-tailed distribution. Then

$$\mathbb{P}(X_1 + X_2 + \cdots + X_n > x) \sim \mathbb{P}\left(\max_{1 \le i \le n}(X_i) > x\right)$$

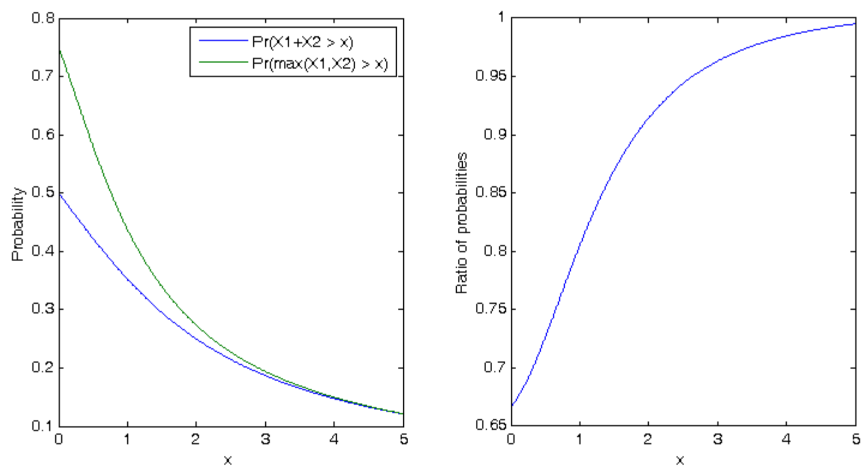

Figure C.2: An illustration of the single big jump phenomena. Left figure show the probability of the sum and maximum of two Cauchy random variables being greater than x and the right figure shows the ratio of these two probabilities.

In figure C.2 we have depicted $\mathbb{P}(X_1 + X_2 > x)$ and $\mathbb{P}(\max_{i=1,2} X_i > x)$ for Cauchy random variables X_1 and X_2. As it is apparent from the figure, the difference between these two probabilities gets negligible very soon. C.2 also shows the ratio between the two probabilities.

We now take a more formal look at this property.

Subexponential Distributions

Let F be a distribution on $\mathbb{R}^+$ with unbounded support. Then F is said to be subexponential if

$$\overline{F * F}(x) \sim 2\bar{F}(x)$$

Now suppose X_1 and X_2 are independent random variables having the same heavy-tailed distribution F. The above definition is equivalent to

$$\mathbb{P}(X_1 + X_2 > x) \sim 2\mathbb{P}(X_1 > x), \quad \text{as} \quad x \to \infty$$

on the other hand we always have

$$\begin{aligned}\mathbb{P}\left(\max_{i=1,2}(X_i) > x\right) &= 1 - \mathbb{P}\left(\max_{i=1,2}(X_i) < x\right) \\ &= 1 - \mathbb{P}(X_1 < x)^2 \\ &= 1 - (1 - \bar{F}(x))^2 \\ &\sim 2\bar{F}(x)\end{aligned}$$

combining there results one can infer the following preposition about the subexponential random variables:

Theorem 7. *If X_1 and X_2 are subexponential random variables then*

$$\mathbb{P}(X_1 + X_2 > x) \sim \mathbb{P}\left(\max_{i=1,2} X_i > x\right), \quad \textit{as} \quad x \to \infty \tag{C.1}$$

This result can be extended to any finite number of random variables.

In order to gain a better understanding of this typical behaviour of the subexponential distributions, we will study the Weibull distribution F_α with parameter α and the tail function $\bar{F}_\alpha$ given by

$$\bar{F}_\alpha = e^{-x^\alpha}, \quad x \geq 0 \tag{C.2}$$

It can be easily shown that the corresponding distribution function would be $f_\alpha(x) = \alpha x^{\alpha-1}e^{-x^\alpha}$ which as already mentioned is heavy-tailed and can be shown to be subexponential if $\alpha < 1$. It is actually a fact that almost all of the natural heavy-tailed distributions, including those previously mentioned ones, are also subexponential. Now let X_1 and X_2 be two independent Weibull distributed random variables having the distribution function given by (C.2). We will examine the probability distribution function of the of the random variable X_1 conditioned on the sum $X_1 + X_2 = c$ for different values of c and the shape parameter α.

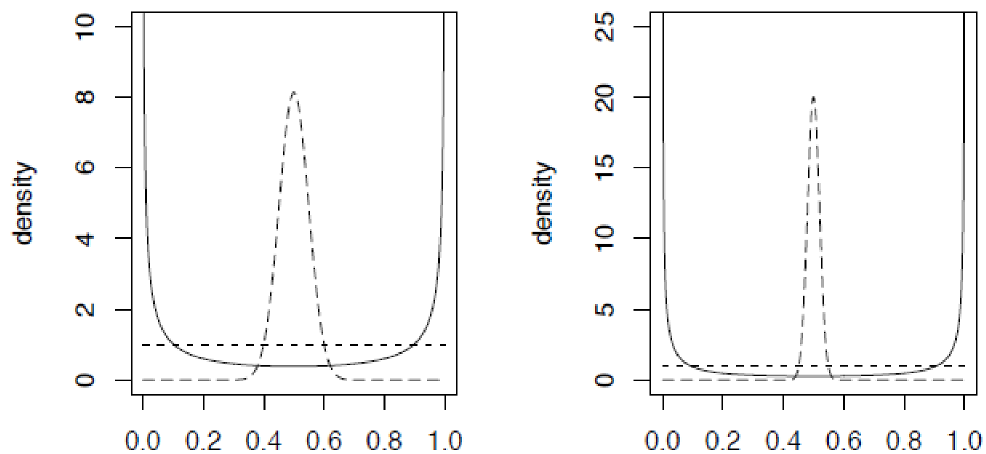

Figure C.3: Probability distribution function of the a Weibull random variables given its sum with another identically distributed Weibull random variable, for different values of the sum and distribution parameter.

The probability density function of the random variable X_1/c given $X_1 + X_2 = c$ is

$$g(z) = a\,(z(1-z))^{\alpha-1}\,e^{-d^{\alpha}(z^{\alpha}+(1-z)^{\alpha})}, \quad 0 \le z \le 1$$

For an appropriate value of the scaler a. Now let us examine this density for different values of α and d. Figure C.2 shows the profile of the density function $g(x)$ for three different values of α and two values of c. The left graph plots $g_{\alpha,c}$ for $c = 10$ and $\alpha = 0.5, 1, 2$. The right hand graph, is the plot of $g_{\alpha,c}$ for $c = 20$ and the same values of α. As it is obvious from each figure, for the case of the heavy-tail distribution, i.e., when $\alpha = 0.5$, the random variable X_1 is most likely to be either equal to 0 or c, both with equal probability, conditioned on the fact that $X_1 + X_2 = c$. And this behaviour is even more apparent as c gets larger, in that the right plot is sharper.

Bibliography

[1] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Information Theory*, vol. 46, no. 2, pp. 388404, March 2000

[2] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran, "Closing the gap in the capacity of wireless networks via percolation theory," *IEEE Trans. Information Theory*, vol. 53, no. 3, pp. 10091018, March 2007.

[3] S. R. Kulkarni and P. Viswanath, "A deterministic approach to throughput scaling in wireless networks," *IEEE Trans. Information Theory*, vol. 50, no. 6, pp. 10411049, June 2004.

[4] S. Toumpis and A. J. Goldsmith, "Large wireless networks under fading, mobility, and delay constraints," *Proc. IEEE Infocom*, vol. 1, Hong Kong, 2004, pp. 609619

[5] M. Gastpar and M. Vetterli, "On the capacity of large gaussian relay networks," *IEEE Trans. Information Theory*, vol. 51, no. 3, pp. 765779, March 2005.

[6] L.-L. Xie and P. R. Kumar, "A network information theory for wireless communication: scaling laws and optimal operation," *IEEE Trans. Information Theory*, vol. 50, no. 5, pp. 748767, May 2004.

[7] S. Webere, J. G. Andrews, and N. Jindal, "Throughput and transmission capacity of ad hoc networks with channel state information," *Allerton Conference on Communication, Control, and Computing, Monticello, IL*, September 2006.

[8] S. Webere, J. G. Andrews, and N. Jindal, "The effect of fading, channel inversion, and threshold scheduling on ad hoc networks," *IEEE Trans. Information Theory*, vol. 53, no. 11, Nvember 2007

[9] R. Gowaikar, B. Hochwald, and B. Hassibi, "Communication over a wireless network with random connections," *IEEE Trans. Information Theory*, vol. 52, no. 7, pp. 28572871, July 2006.

[10] R. Gowaikar and B. Hassibi, "On the achievable throughput in two-scale wireless networks," *Proc. IEEE ISIT*, 2006

[11] D. Gesbert and M. Kountouris, "Resource allocation in multicell wireless networks: Some capacity scaling laws," *Proc. Workshop on Resource Allocation in Wireless NETworks (RAWNET 07)*, 2007.

[12] D. Gesbert and M. Kountouris, "Rate scaling laws in multicell networks under distributed power control and user scheduling" *IEEE Trans. Inf. theory,* vol. 57, no. 1, pp 234-244, Jan. 2011.

[13] T.M Cover and J. A. Thomas,*Elements of Information Theory.* New York: John Wiley & Sons, Inc., 1991.

[14] A. Gut, *Probability: A Graduate Course,* Springer, 2005

[15] Paul Embrechts, Claudia Klppelberg and Thomas Mikosch, *Modelling Extremal Events for Insurance and Finance*,Springer, 1997